Exercise 9

By factoring $z^4 - 4z^2 + 3$ into two quadratic factors and using inequality (2), Sec. 5, show that if z lies on the circle $|z| = 2$, then

$$\left|\frac{1}{z^4 - 4z^2 + 3}\right| \leq \frac{1}{3}.$$

Solution

$$\begin{aligned}
\left|\frac{1}{z^4 - 4z^2 + 3}\right| &= \frac{1}{|z^4 - 4z^2 + 3|} \\
&= \frac{1}{|(z^2 - 1)(z^2 - 3)|} \\
&= \frac{1}{|z^2 - 1||z^2 - 3|} \\
&\leq \frac{1}{||z^2| - |1||||z^2| - |3||} \\
&= \frac{1}{||z|^2 - 1|||z|^2 - 3|} \\
&= \frac{1}{|2^2 - 1||2^2 - 3|} \\
&= \frac{1}{(3)(1)} \\
&= \frac{1}{3}
\end{aligned}$$

Therefore, if $|z| = 2$, then

$$\left|\frac{1}{z^4 - 4z^2 + 3}\right| \leq \frac{1}{3}.$$